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Final Report on

EFFECT OF NONLINEAR ARCHITECTURES ON MONOPULSE RADAR ANGLE TRACKING ACCURACY

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FORWARD

This report describes work done within the Department of Electrical Engineering of the University of Florida, Gainesville, Florida 32611, for the U. S. Naval Research Laboratory, Washington, D. C. 20375 under contract N00014-82-K-2008, during the period 1 December 1981 to 1 December 1982.

Dr. Peyton Z. Peebles, Jr. was the Principal Investigator and he performed all of the theoretical analyses. Mr. Derick Chen conducted the accuracy evaluations reported in Section 3 that relate to rectangular transmitter pulse waveform shape. The final report was prepared by Dr. Peebles.

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ABSTRACT

Many monopulse angle tracking radar systems are made up using nonlinear components, such as limiters and logarithmic amplifiers. This report describes several areas of work that all relate to monopulse angle tracking radars having at least one limiter in the signal processor. In one area the detailed angle tracking accuracy was determined for a sum-and-difference amplitude monopulse system having a sum channel limiter prior to the angle error detector. The analysis utilized a general accuracy expression valid for an arbitrary angle tracking system; the expression was developed as a second area of effort. A third area of work centered on evaluating coefficients that enter into the accuracy results derived in the first-described area. A fourth work area involved computation of the accuracy of a Rhodes Class III monopulse (with two limiters); this work was incomplete at the project end and remains for future work.

CONTENTS

1.0 INTRODUCTION

Accuracy of angle tracking radar systems when noise is present has been the subject of many analyses. Until the earlier phase of this study [1][†] most work did not employ the rigorous analytical methods currently available. In [1] rigorous methods were used to develop an accuracy formula for the mean-squared angle tracking error of four monopulse systems. These four systems could be categorized as "linear" because their architectures did not involve nonlinear signal processing components, such as limiters or logarithmic amplifiers.

It is the purpose of this report to describe the results of a rigorous study of monopulse angle tracking accuracy applicable to systems with nonlinear elements. The reported results apply only to architectures having a limiter or limiters present. Architectures having logarithmic amplifies (such as the Rhodes class II system [2]) constitute an area of possible future research.

This report describes work that falls mainly in four specific areas. First, the tracking accuracy is found for a sum-and-difference channel form of amplitude monopulse system when the sum channel contains limiting; this is probably one of the most popular architectures in use today. The limiting can occur because the sum channel input to the angle error (phase) detector is saturated, or because a separate limiter is used prior to the angle detector. In either case the effect can be modeled as a limiter preceeding the angle error detector. This type of system, which will be called system 5, is analyzed in Section 3.0.

The second area of analysis concerns a general expression for

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† References are cited by bracketed number and are listed at the end of the report.

the angle accuracy of an arbitrary system, which includes those having nonlinear elements. The resulting expression formed the basis for the accuracy results derived for system 5.

The third area of work involved the numerical evaluation of some coefficients that appear in the accuracy expressions of system 5. This work was only partially completed at the end of the study because of some computer programming difficulties.

The fourth major area of effort centered around the accuracy analysis of a Rhodes Class III monopulse angle tracking system (one with two limiters). This work was partially completed and is expected to be completed through future effort.

Finally, a number of other minor tasks were undertaken, mostly to support the above mentioned four efforts. The most significant other task involved extending the general analysis of Kelly and Hariharan [3] for the general response of an ideal limiter when its input is the sum of a bandpass periodic signal and bandpass Gaussian noise.

2.0 GENERAL THEORY FOR ACCURACY OF ANGLE TRACKING RADAR

2.1 Introduction

In all angle tracking radars there is some point where an "error" voltage is developed that is used to force the antenna patterns to follow, or track, the target. There is one such error voltage developed for each angle coordinate. The error voltages may be the result of a "linear" system, such as a sum-and-difference monopulse processor having a "linear" error detector or they can result from a nonlinear system. A commonly used nonlinear system is the ordinary sum-and-difference processor where the error detector (a phase detector) is operated with one input (the sum channel input) saturated. Another example would be a Rhodes' class III [2] monopulse receiver having a limiter in each channel.

Regardless of how the error voltage is developed it contains all the information available to the tracking system about the target's location. Since it also contains noise, its use in estimating (tracking) target angle will have errors. In the following section we develop a general equation for the mean-squared angle tracking error (variance for a zero-bias system) of a monopulse radar that is valid even if the system contains nonlinear elements in its signal processor.

2.2 Accuracy Expression

In Figure 2-1 let v represent the output of some monopulse angle tracking system that is to be used by an angle servo to track a target in an angle coordinate denoted by x. Because angle dynamics are usually small in most systems, there is usually some form of lowpass filtering (LPF) of v prior to its use. Let u represent the filter's output. We shall assume the filter's bandwidth to be small relative to the system's

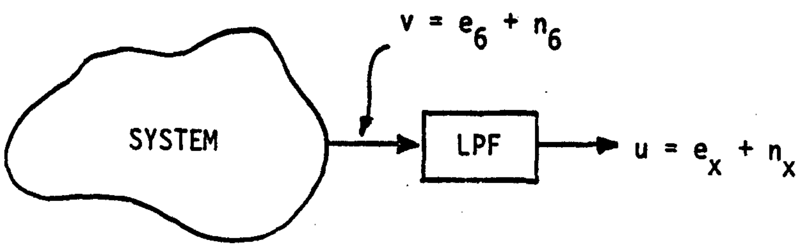

Fig. 2-1. Arbitrary system generating an angle error voltage.

pulse rate (PRF) but otherwise can be as large as necessary.

The angle tracker can operate only on the "signal" part of u. Because noise is present we could take its effect into account by defining the signal as the average value of u E(U). [Here E(·) denotes the statistical average taken over the noise modeled as a sample function of a random process U(t)]. This definition is only partially acceptable because E(U) is still a function of time. It is therefore the time average of E(U), denoted by $\mathcal{A}\{E(U)\}$, that defines the "signal" component of u. Thus,

$$e_x \triangleq \mathcal{A}\{E(U)\} \tag{2.2-1}$$

$$n_x = u - e_x = u - \mathcal{A}\{E(U)\}. \tag{2.2-2}$$

If the noise $n_x(t)$ is modeled as a sample function of a random process N(t), the time averaged autocorrelation function of the noise process will reduce to

$$\begin{aligned} R_N(\tau) &\triangleq \mathcal{A}\{R_N(t, t+\tau)\} = \mathcal{A}\{E\,[N(t)\,N(t+\tau)]\} \\ &= \mathcal{A}\{R_U(t, t+\tau)\} - \mathcal{A}\{E[U(t)]\}\,\mathcal{A}\{E[U(t+\tau]\}. \end{aligned} \tag{2.2-3}$$

Next, let x represent the coordinate of possible target angles (relative to boresight) and let x_0 be a specific value. For values of x near x_0

$$e_x(x) \approx e_x(x_0) + \left.\frac{de_x}{dx}\right|_{x=x_0} (x - x_0). \tag{2.2-4}$$

An error-free measurement of u would equal $e_x(x_0)$. With noise we have an arbitrary u and $e_x(x_0)$ is replaced by $e_x(x)$ of (2.2-4) where $(x-x_0)$ is interpreted as the measurement error in x_0 that is associated with the noisy value of u. Hence,

$$u = e_x(x_o) + \frac{de_x}{dx}\bigg|_{x=x_o} (x - x_o) \qquad (2.2\text{-}5)$$

so

$$u - e_x(x_o) = n_x = \frac{de_x}{dx}\bigg|_{x=x_o} (x - x_o) \, . \qquad (2.2\text{-}6)$$

The mean-squared error in angle is defined as

$$\sigma^2(\hat{x}) \triangleq \mathcal{A}\{E\,[(x-x_o)^2]\}. \qquad (2.2\text{-}7)$$

From (2.2-6) using (2.2-7) and (2.2-3) with $\tau=0$:

$$\sigma^2(\hat{x}) = \frac{\mathcal{A}\{E[x^2(t)]\}}{\left[\frac{de_x}{dx}\Big|_{x=x_o}\right]^2} = \frac{\mathcal{A}\{R_x(t,t)\}}{\left[\frac{de_x}{dx}\Big|_{x=x_o}\right]^2} \qquad (2.2\text{-}8)$$

or

$$\sigma^2(\hat{x}) = \frac{\mathcal{A}\{R_U(t,t)\} - \left\langle \mathcal{A}\{E[U(t)]\}\right\rangle^2}{\left[\frac{de_x}{dx}\Big|_{x=x_o}\right]^2} \, . \qquad (2.2\text{-}9)$$

Usually we are most interested in accuracy near boresight where $x = x_o = 0$.

Other forms can be obtained for (2.2-9) which is a basic result applicable to nearly any system. If we let $\mathcal{S}_U(\omega)$ and $\mathcal{S}_V(\omega)$ denote the Fourier transforms of $R_U(\tau)$ and $R_V(\tau)$, the time averaged autocorrelation of $v(t)$, then it can be shown [4], p. 450, that

$$\mathcal{S}_U(\omega) = \mathcal{S}_V(\omega) \mid H(\omega) \mid^2 \qquad (2.2\text{-}10)$$

where $H(\omega)$ is the transfer function of the LPF. Thus,

$$\mathcal{A}\{R_U(t,t)\} \cdot R_U(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_V(\omega)|\,H(\omega)|^2\, d\omega$$

$$\approx \frac{1}{2\pi}\int_{-W_L}^{W_L} \mathcal{S}_V(0)\, d\omega = \frac{W_L}{\pi}\int_{-\infty}^{\infty} R_V(\tau)\, d\tau$$

$$= \frac{W_L}{\pi}\int_{-\infty}^{\infty} \mathcal{A}\{R_V(t, t+\tau)\}\, d\tau \qquad (2.2\text{-}11)$$

where W_L (rad/s) is the noise bandwidth of the LPF. By using (2.2-11) with (2.2-9) we have

$$\boxed{\sigma^2(\hat{x}) \approx \frac{\frac{W_L}{\pi}\int_{-\infty}^{\infty}\mathcal{A}\{R_V(t, t+\tau)\}\, d\tau - \left\langle \mathcal{A}\{E[V(t)]\}\right\rangle^2}{\left\langle \frac{d\ \mathcal{A}\{E[V(t)]\}}{dx}\Big|_{x=x_0}\right\rangle^2}\ .} \qquad (2.2\text{-}12)$$

In writing (2.2-12) use has been made of

$$\mathcal{A}\{E[U(t)]\} = \mathcal{A}\{E[V(t)]\}. \qquad (2.2\text{-}13)$$

Finally, we observe that in many systems $e_x = \mathcal{A}\{E[U(t)]\}$ is an odd function of x. When this case is true (2.2-12) will reduce to

$$\sigma^2(\hat{x}) = \frac{\frac{W_L}{\pi}\int_{-\infty}^{\infty}\mathcal{A}\{R_V(t,t+\tau)\}d\tau}{\left\langle \frac{d\ \mathcal{A}\{E[V(t)]\}}{dx}\Big|_{x=x_0=0}\right\rangle^2} \text{ ; } \mathcal{A}\{E[V(t)]\} \text{ is odd function of x.} \qquad (2.2\text{-}14)$$

3.0 ANGLE ACCURACY OF A MONOPULSE TRACKING RADAR HAVING A SUM-CHANNEL LIMITER

3.1 INTRODUCTION

In this section we shall find the angle tracking accuracy of a sum-and-difference form of amplitude comparison monopulse radar when the sum channel contains a limiter. Accuracy will be determined through the mean-squared angle error (same as error variance when the system has no bias, as is the case here). Thus, we shall basically seek the solution of (2.2-14) for the system of interest.

Figure 3-1 illustrates the block diagram of the system, which we call System 5 (four systems were previously studied in [1]). Some functions have been omitted from Fig. 3-1 because they do not affect the accuracy results. For example, the down-converting mixers usually present in both channels only affect the frequency at which the components operate but do not affect signal forms (shape, bandwidth, etc.) so do not affect accuracy. In a similar manner the automatic gain control (AGC) loop usually has a range-gated envelope detector; we shall assume the AGC loop works well enough that it does not affect accuracy and these gate/detector functions can be omitted.

In Fig. 3-1 we make several assumptions. The noises that are present at the inputs to the matched filters (MF's) are assumed to be white, stationary, zero-mean and Gaussian with equal power density spectra $\mathcal{N}_0/2$. The amplifiers (AMP's) are assumed to be identical in gain, are affected by the AGC loop in the same manner and any band-shaping they might contribute to the channels in practice is accounted for in the transfer function of the matched filters. Thus, the noises n_2 and n_4 are Gaussian, bandpass, and zero-mean with identical power

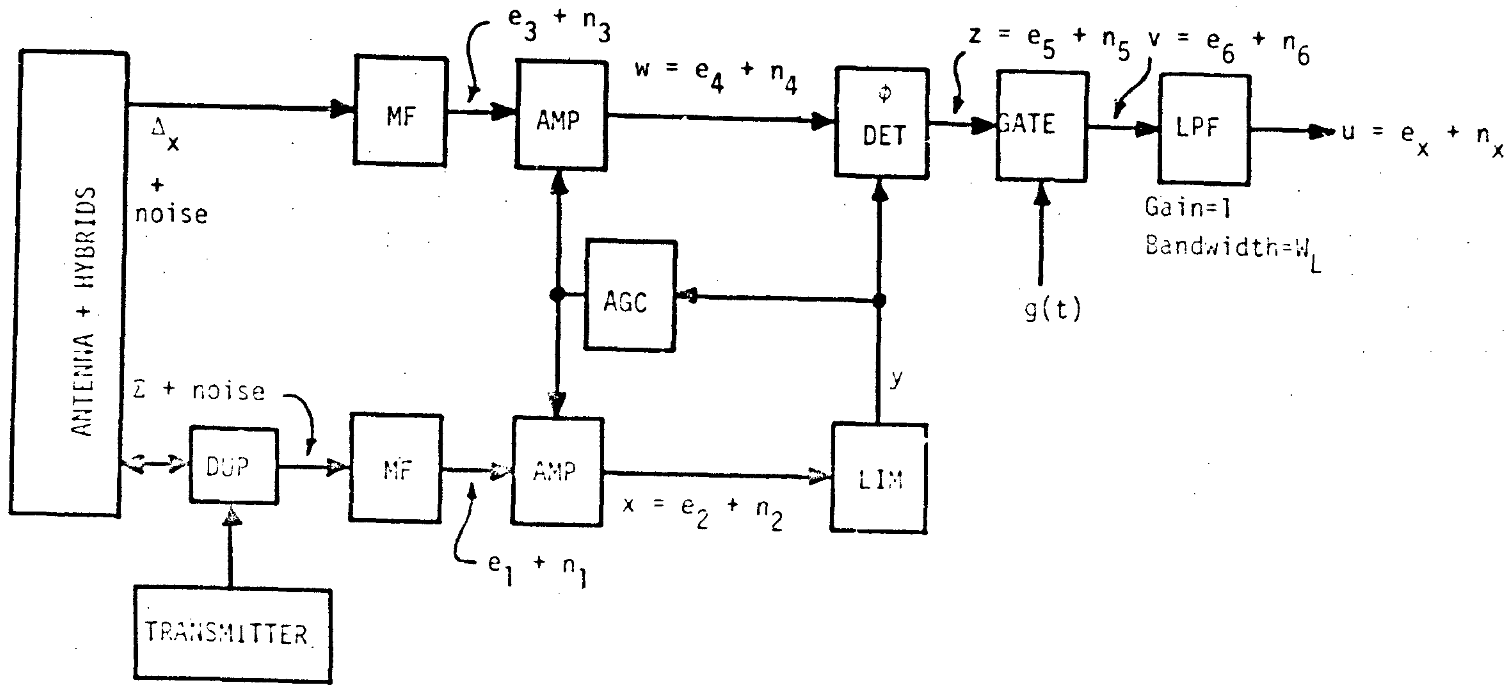

Fig. 3-1. Block diagram of the sum and one error channel of System E.

spectra. Their powers are also equal and denoted by σ^2.

The above assumptions are identical to those made in an earlier study [1] for System 1. In fact, both signals and noises are identical up to the outputs of the MF's. From [1] we have

$$e_2(t) = \frac{V_\Sigma}{c(0)} \sum_{n=-\infty}^{\infty} c(t-\tau_T-n\,T_R)\cos\left[\omega_o(t-\tau_T)+\theta_o\right] \qquad (3.1\text{-}1)$$

$$e_4(t) = \frac{V_\Sigma\, k_m\, x}{c(0)\,\theta_3} \sum_{n=-\infty}^{\infty} c(t-\tau_T-nT_R)\cos\left[\omega_o(t-\tau_T)+\theta_o\right] \qquad (3.1\text{-}2)$$

$$c(t) = \int_{-\infty}^{\infty} p(\xi)\,p(\xi+t)\,d\xi. \qquad (3.1\text{-}3)$$

Here V_Σ is the peak amplitude of the sum channel pulses at the input to the limiter (LIM); the shape of these pulses is determined by $c(t)$ which is related to the shape of transmitted pulses determined by $p(t)$. The parameter τ_T is the two-way target range delay, T_R is the system repetition period (pulse repetition rate is $f_R = 1/T_R$), ω_o is the radar's carrier angular frequency, θ_o is an arbitrary phase angle, x is the target's error angle from the boresight axis in the coordinate of interest,*

$$k_m \triangleq \frac{\theta_3}{G_\Sigma} \left.\frac{\partial G_\Delta(x)}{\partial x}\right|_{x=0}, \qquad (3.1\text{-}4)$$

θ_3 is the beamwidth of the one-way antenna sum pattern between -3dB points in the x coordinate direction, $G_\Delta(x)$ is the one-way x-coordinate antenna voltage difference pattern, and G_Σ is the one-way voltage gain of the sum pattern on the boresight axis.

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* Accuracy is developed only for the x coordinate. An analysis for the space-orthogonal coordinate would be identical in form. Numerical results would be different in the two coordinates only if parameters (θ_3, k_m, etc.) are different in the two coordinate directions.

Noises n_2 and n_4 have equal power spectra, as given in [1], that can be put in the form

$$\mathcal{S}_{N_2}(\omega) = \mathcal{S}_{N_4}(\omega) = \frac{\sigma^2}{2c(0)} \left[|P(\omega - \omega_0)|^2 + |P(\omega + \omega_0)|^2 \right], \qquad (3.1\text{-}5)$$

where

$$c(t) \longleftrightarrow |P(\omega)|^2; \qquad (3.1\text{-}6)$$

$P(\omega)$ denotes the Fourier transform of $p(t)$, and the double-ended arrow represents a Fourier transform pair [transform of $c(t)$ is $|P(\omega)|^2$].

It is clear from (3.1-1) that the single-pulse peak signal power to average noise power ratio of the waveform at the input to the limiter, denoted by (S_1/N), is

$$(S_1/N) = V_\Sigma^2 / 2\sigma^2. \qquad (3.1\text{-}7)$$

The relationships derived or stated in this section form the link with the earlier study and define most of the quantities common to both studies. To arrive at the desired system accuracy we must now compute the mean value and autocorrelation function of the system's response [$v(t)$ in Figure 3-1].

3.2 Mean Value of System Response

From Figure 3-1 we have

$$v(t) = g(t)\, z(t) = g(t)\, w(t)\, y(t). \qquad (3.2\text{-}1)$$

Since $w(t) = e_4(t) + n_4(t)$ and $n_4(t)$ is statistically independent of $n_2(t)$, and therefore independent of $y(t)$, the mean value of the process $V(t)$, having $v(t)$ as a sample function, is

$$E[V(t)] = g(t)\, E[W(t)] E[Y(t)] = g(t)\, e_4(t)\, E\,[Y(t)]. \qquad (3.2\text{-}2)$$

To evaluate $E[Y(t)]$ we note that the limiter is described by its transfer characteristic $g(X)$ where

$$Y = g(X) = \frac{X}{|X|} = \begin{cases} 1, & X>0 \\ -1, & X<0. \end{cases} \tag{3.2-3}$$

Thus, if $f_X(x)$ is the probability density function of the limiter's input process $X(t)$ and $f_{N_2}(n_2)$ is the density of the process $N_2(t)$ representing the noise $n_2(t)$ we have

$$E[Y] = E\left[\frac{X}{|X|}\right] = \int_{-\infty}^{\infty} \frac{x}{|x|} f_X(x)\, dx = \int_{-\infty}^{\infty} \frac{x}{|x|} f_{N_2}(x - e_2)\, dx$$

$$= \int_0^{\infty} f_{N_2}(x - e_2)\, dx - \int_{-\infty}^{0} f_{N_2}(x - e_2)\, dx. \tag{3.2-4}$$

since

$$f_X(x) = f_{N_2}(x - e_2). \tag{3.2-5}$$

By substituting for $f_{N_2}(\cdot)$ which is a zero-mean Gaussian density with variance σ^2, and using simple variable transformations (3.2-5) will reduce to

$$E[Y] = \operatorname{erf}\left(\frac{e_2}{\sqrt{2}\,\sigma}\right) \tag{3.2-6}$$

where erf $(\cdot)$ is the error function defined by

$$\operatorname{erf}(\alpha) \triangleq \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-\xi^2}\, d\xi. \tag{3.2-7}$$

It is also known that

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!\,(2k+1)}. \tag{3.2-8}$$

By using (3.2-8) and (3.2-6) in (3.2-2) we have

$$E[V(t)] = g(t)\ e_4(t)\ \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \left(\frac{e_2}{\sqrt{2}\ \sigma}\right)^{2k+1} . \tag{3.2-9}$$

3.3 Time-Averaged Mean Value

By defining

$$Per(t) \stackrel{\Delta}{=} \sum_{n=-\infty}^{\infty} c(t - \tau_T - nT_R) \tag{3.3-1}$$

$$\theta \stackrel{\Delta}{=} \omega_0(t - \tau_T) - n\ T_R \tag{3.3-2}$$

and substituting these as well as (3.1-7), (3.1-1) and (3.1-2) into (3.2-9), the time average of (3.2-9) can be written

$$\mathcal{A}\{E[V]\} = \frac{2\sqrt{2}\ k_m \times \sigma}{\theta_3 \sqrt{\pi}} \left(\frac{S_1}{N}\right) \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{k!(k+1)}\right]$$

$$\cdot \mathcal{A}\left\{g(t) \left[\frac{Per(t)}{c(0)}\right]^{2k+2} \cos^{2k+2}(\theta)\right\} . \tag{3.3-3}$$

Because

$$\cos^{2k+2}(\theta) = \frac{1}{2^{2k+2}} \left\{ \binom{2k+2}{k+1} + 2 \sum_{m=0}^{k} \binom{2k+2}{m} \cos[(2k+2-2m)\theta] \right\} \tag{3.3-4}$$

the typical term in (3.3-2) is the time average of the product of a lowpass function and a bandpass function. Such terms will be nearly zero because cos $[2(k+1-m)\theta]$ cycles rapidly during the times the low-pass terms exist. Hence, only the constant term in (3.3-4) gives rise to a nonzero result in (3.3-3). The remaining time averages are of a periodic function (period = T_R). With the definitions

$$\boxed{\bar{d}_{k+1} \triangleq \int_{-T_R/2}^{T_R/2} \left[\frac{c(t)}{c(0)}\right]^{2k+2} \mathrm{rect}\left(\frac{t}{\tau_g}\right) \, dt} \tag{3.3-5}$$

$$\mathrm{rect}\left(\frac{t}{\tau_g}\right) \triangleq \begin{cases} 1, & |t| < \tau_g/2 \\ 0, & |t| > \tau_g/2 \end{cases} \tag{3.3-6}$$

$$\boxed{R_{ck} \triangleq \frac{(-1)^k \, (2k)!}{(k!)^2 \, (k+1)! \; 2^{2k}} \left(\frac{\bar{d}_{k+1}}{\bar{d}_1}\right) ,} \tag{3.3-7}$$

we can finally write (3.3-3) as

$$\mathcal{A}\{E[V]\} = \frac{k_m \times V_\Sigma \sqrt{S_1/N} \quad \bar{d}_1}{\theta_3 \sqrt{\pi} \; T_R} \sum_{k=0}^{\infty} R_{ck} \left(\frac{S_1}{N}\right)^k . \tag{3.3-8}$$

Next, if we recognize

$$\sigma_{min}^2(\hat{x}) = \frac{\theta_3^2}{2k_m^2 \, (S_1/N)} \cdot \frac{W_L T_R}{\pi} \tag{3.3-9}$$

as the smallest angle error variance possible for any monopulse system [1], the derivative required in (2.2-14) is

$$\left.\frac{d\mathcal{A}\{E[V]\}}{dx}\right|_{x=0} = \frac{V_\Sigma \, \bar{d}_1 \sqrt{W_L}}{\left[2\pi^2 \sigma_{min}^2(\hat{x}) \; T_R\right]^{1/2}} \sum_{k=0}^{\infty} R_{ck} \left(\frac{S_1}{N}\right)^k . \tag{3.3-10}$$

As an intermediate point in computing overall system accuracy we use (3.3-10) in (2.2-14) to write

$$\sigma^2(\hat{x}) = \frac{\sigma_{min}^2(\hat{x}) \, \dfrac{2\pi \, T_R}{V_\Sigma^2 \, \bar{d}_1^2} \displaystyle\int_{-\infty}^{\infty} \mathcal{A}\{R_V(t, t+\tau)\} \, d\tau}{\left\{ \displaystyle\sum_{k=0}^{\infty} R_{ck} \left(\frac{S_1}{N}\right)^k \right\}^2} . \tag{3.3-11}$$

3.4 Autocorrelation of System Response

To complete the solution of (3.3-11) we must find $R_V(t, t+\tau)$. From (3.2-1)

$$R_V(t,t+\tau) = E\,[V(t)\,V(t+\tau)] = g(t)g(t+\tau)\,E[W(t)W(t+\tau)\,Y(t)Y(t+\tau)]$$

$$= g(t)g(t+\tau)\,E[W(t)W(t+\tau)]\,R_Y(t,\,t+\tau). \tag{3.4-1}$$

Since $W(t) = e_4(t) + N_4(t)$ and the noise is zero mean, (3.4-1) reduces to

$$R_V(t,t+\tau) = g(t)g(t+\tau)\,R_{N_4}(\tau)\,R_Y(t,t+\tau) \tag{3.4-2}$$

where $R_{N_4}(\tau)$ is the autocorrelation function of the stationary noise process $N_4(t)$. In writing (3.4-2) we have used the approximation $e_4(t) \approx 0$ for near-boresight tracking.

By inverse transformation of (3.1-5), $R_{N_4}(\tau)$ will be

$$R_{N_4}(\tau) = \sigma^2\,\frac{c(\tau)}{c(0)}\,\cos(\omega_0\tau) = \sigma^2\,\frac{c(\tau)}{c(0)}\,\cos\,(\theta_1) \tag{3.4-3}$$

where we define

$$\theta_1 \triangleq \omega_0\,\tau. \tag{3.4-4}$$

Thus,

$$R_V(t,t+\tau) = g(t)g(t+\tau)\,\sigma^2\,\frac{c(\tau)}{c(0)}\,\cos\,(\theta_1)\,R_Y(t,t+\tau). \tag{3.4-5}$$

Our main remaining task is to solve for the autocorrelation function $R_Y(\cdot,\cdot)$.

If we define the joint probability density function of the process $X(t)$ at times t and $t+\tau$ as $f_X(x_1,x_2)$, where $X_1 \triangleq X(t)$ and $X_2 \triangleq X(t+\tau)$, then the responses are denoted $Y_1 \triangleq Y(t)$ and $Y_2 \triangleq Y(t+\tau)$,

and

$$R_Y(t,t+\tau) = E[Y(t)Y(t+\tau)] = E[Y_1Y_2] = E\left[\frac{X(t)}{|X(t)|}\frac{X(t+\tau)}{|X(t+\tau)|}\right]$$

$$= E\left[\frac{X_1}{|X_1|}\frac{X_2}{|X_2|}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x_1}{|x_1|}\frac{x_2}{|x_2|} f_X(x_1,x_2)\, dx_1\, dx_2. \qquad (3.4\text{-}6)$$

Next, we note that in the x_1 x_2 plane

$$\frac{x_1}{|x_1|}\frac{x_2}{|x_2|} = \begin{cases} 1, & \text{quadrants 1 and 3} \\ -1, & \text{quadrants 2 and 4.} \end{cases} \qquad (3.4\text{-}7)$$

Furthermore,

$$f_X(x_1,x_2) = f_{N_2}(x_1 - e_1, x_2 - e_2) \qquad (3.4\text{-}8)$$

where $e_1 \overset{\Delta}{=} e_2(t)$ and $e_2 \overset{\Delta}{=} e_2(t+\tau)$ and $f_{N_2}(\cdot,\cdot)$ is the joint density of noise process $N_2(t)$ at times t and $t + \tau$. Because these noises are Gaussian

$$f_{N_2}(\alpha,\beta) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{2\sigma^2(1-\rho^2)}\right\} \qquad (3.4\text{-}9)$$

where ρ is the correlation coefficient given by

$$\rho = \sigma^{-2} R_{N_2}(\tau). \qquad (3.4\text{-}10)$$

By using (3.4-7) through (3.4-10) in (3.4-6) it can be shown that

$$R_Y(t,t+\tau) = -1 + 2\left[P\left(\frac{e_2}{\sigma}\right) - Q\left(\frac{e_1}{\sigma}\right)\right] + 4L\left[\frac{e_1}{\sigma}, \frac{e_2}{\sigma}, \rho\right], \qquad (3.4\text{-}11)$$

where [5, p. 936]

$$P(k) \overset{\Delta}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k} e^{-\xi^2/2}\, d\xi = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\,(k/\sqrt{2}) \qquad (3.4\text{-}12)$$

$$Q(k) \overset{\Delta}{=} \frac{1}{\sqrt{2\pi}} \int_k^\infty e^{-\xi^2/2}\, d\xi = \frac{1}{2} - \frac{1}{2} \operatorname{erf}(k/\sqrt{2}), \tag{3.4-13}$$

and

$$L(h, k, \rho) \overset{\Delta}{=} \int_{y=k}^\infty \int_{x=h}^\infty g(x, y, \rho)\, dx\, dy, \tag{3.4-14}$$

where [5, p. 936]

$$g(x,y,\rho) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left\{- \frac{x^2 - 2\rho\, xy + y^2}{2\,(1 - \rho^2)}\right\}. \tag{3.4-15}$$

From known results [5, p. 934, eq. 26.2.32 and p. 940, eq. 26.3.29] it can be shown that (3.4-11) will reduce to

$$R_Y(t,t + \tau) = \operatorname{erf}\left(\frac{e_1}{\sqrt{2}\,\sigma}\right) \operatorname{erf}\left(\frac{e_2}{\sqrt{2}\,\sigma}\right)$$

$$+4 \sum_{n=0}^\infty \frac{\rho^{\,n+1}}{(n+1)!\; 2^n} H_n\left(\frac{e_1}{\sqrt{2}\,\sigma}\right) H_n\left(\frac{e_2}{\sqrt{2}\,\sigma}\right) Z\left(\frac{e_1}{\sigma}\right)$$

$$\cdot Z\left(\frac{e_2}{\sigma}\right) \tag{3.4-16}$$

where

$$Z(\xi) \overset{\Delta}{=} \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2} = \frac{1}{\sqrt{2\pi}} \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!} \left(\frac{\xi^2}{2}\right)^\ell \tag{3.4-17}$$

and $H_n(\xi)$ is the Hermite polynomial given by

$$H_n(\xi) = n! \sum_{m=0}^{[n]/2} \frac{(-1)^m\, 2^{n-2m}\, x^{n-2m}}{m!\; (n - 2m)!}. \tag{3.4-18}$$

Here [n] is the largest even integer not exceeding n.

We separately expand the terms in (3.4-16). From use of (3.2-8) we expand the first term as

$$\operatorname{erf}\left(\frac{e_1}{\sqrt{2}\,\sigma}\right)\operatorname{erf}\left(\frac{e_2}{\sqrt{2}\,\sigma}\right) = \frac{4}{\pi}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}$$

$$\frac{(-1)^{i+k}\, e_2^{2i+1}(t)\, e_2^{2k+1}(t+\tau)}{(2i+1)\; i!\; (2k+1)\; k!\; (2\sigma^2)^{i+k+1}} \,. \tag{3.4-19}$$

If (3.4-17) and (3.4-18) are used in the second term of (3.4-16) we have

$$4\sum_{n=0}^{\infty}\frac{\rho^{n+1}}{(n+1)!\;2^n} H_n\left(\frac{e_1}{\sqrt{2}\,\sigma}\right) H_n\left(\frac{e_2}{\sqrt{2}\,\sigma}\right) Z\left(\frac{e_1}{\sigma}\right) Z\left(\frac{e_2}{\sigma}\right)$$

$$= 4\sum_{n=0}^{\infty}\sum_{m=0}^{[n]/2}\sum_{\ell=0}^{[n]/2}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{E_{nm\ell}\;\rho^{n+1}(\tau)[e_2(t)]^{n-2m+2r}}{(\sigma^2)^{n-m-\ell+r+p}}$$

$$\cdot\,[e_2(t+\tau)]^{n-2\ell+2p} \tag{3.4-20}$$

where

$$E_{nm\ell} \stackrel{\Delta}{=} \frac{(-1/2)^{m+\ell+r+p}\,(n!)^2}{2\pi\;(n+1)!\;m!\;(n-2m)!\;\ell!\;(n-2\ell)!\;r!\;p!} \,. \tag{3.4-21}$$

Next, we use (3.3-1) and (3.3-2) to write (3.1-1) as

$$e_2(t) = \frac{V_\Sigma}{c(0)}\,\mathrm{Per}(t)\cos(\theta). \tag{3.4-22}$$

Therefore,

$$e_2(t+\tau) = \frac{V_\Sigma}{c(0)}\,\mathrm{Per}(t+\tau)\cos(\theta+\theta_1) \tag{3.4-23}$$

where (3.4-4) has been substituted. After using these two expressions with (3.4-20), (3.4-19) and (3.1-7), we can write (3.4-16) as

$$R_Y(t,t+\tau) = \frac{4}{\pi}\left(\frac{S_1}{N}\right)\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-S_1/N)^{i+k}}{(2i+1)\ i!\ (2k+1)k!}$$

$$\cdot\left[\frac{Per(t)}{c(0)}\right]^{2i+1}\left[\frac{Per(t+\tau)}{c(0)}\right]^{2k+1}\left[\cos(\theta)\right]^{2i+1}\left[\cos(\theta+\theta_1)\right]^{2k+1}$$

$$+\ 4\sum_{n=0}^{\infty}\sum_{m=0}^{[n]/2}\sum_{\ell=0}^{[n]/2}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}E_{nm\ell}\left(2\,\frac{S_1}{N}\right)^{n-m-\ell+r+p}$$

$$\cdot\left[\frac{c(\tau)}{c(0)}\right]^{n+1}\left[\frac{Per(t)}{c(0)}\right]^{n-2m+2r}\left[\frac{Per(t+\tau)}{c(0)}\right]^{n-2\ell+2p}$$

$$\cdot\left[\cos(\theta_1)\right]^{n+1}\left[\cos(\theta)\right]^{n-2m+2r}\left[\cos(\theta+\theta_1)\right]^{n-2\ell+2p}\ . \quad (3.4\text{-}24)$$

3.5 Time-Averaged Autocorrelation

Our principal goal is to solve (3.3-11) which requires that we find $\mathcal{A}\{R_V(t, t+\tau)\}$ by time-averaging (3.4-5) using (3.4-24). We proceed first with the double-sum term in (3.4-24). First we substitute the term into (3.4-5) and form the time average. By grouping terms involving time t we observe that $\cos^{2i+1}(\theta)\cos^{2k+1}(\theta+\theta_1)$ is periodic with period T_R, and cycles rapidly with t [recall that θ is a function of t through (3.3-2)] while other factors vary slowly (are baseband). By Fourier series expansion of $\cos^{2i+1}(\theta)\cdot\cos^{2k+1}(\theta+\theta_1)$ only the constant term (in t) leads to a baseband result which is unfiltered by the output lowpass filter. The time average then equals this constant term times the time average of other baseband time terms. The overall result is a double sum of terms that are periodic in the variable θ_1.

Again, a Fourier series expansion is used and only the constant

(baseband) terms are preserved. The final result becomes

$$\text{First part of } \frac{2\pi\, T_R}{V_\Sigma^2\, \bar{d}_1^{\,2}} \int_{-\infty}^{\infty} \mathcal{A}\{R_V(t,t+\tau)\}\, d\tau$$

$$= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} H_{ik}\, \bar{R}_{Aik} \left(\frac{S_1}{N}\right)^{i+k}, \tag{3.5-1}$$

where

$$\boxed{H_{ik} \overset{\Delta}{=} \frac{(-1)^{i+k}}{(2i+1)\, i!\, (2k+1)\, k!} \left(\frac{1}{2\pi}\right)^2 \cdot \int_{-\pi}^{\pi} \cos(\theta_1) \int_{-\pi}^{\pi} \cos^{2i+1}(\theta) \cos^{2k+1}(\theta+\theta_1)\, d\theta\, d\theta_1} \tag{3.5-2}$$

$$\boxed{\bar{R}_{Aik} \overset{\Delta}{=} \frac{\int_{-\infty}^{\infty} \frac{c(\tau)}{c(0)} \int_{-T_R/2}^{T_R/2} \text{rect}\left(\frac{t}{\tau_g}\right)\left[\frac{c(t)}{c(0)}\right]^{2i+1} \text{rect}\left(\frac{t+\tau}{\tau_g}\right)\left[\frac{c(t+\tau)}{c(0)}\right]^{2k+1} dt\, d\tau}{\left\{\int_{-T_R/2}^{T_R/2} \text{rect}\left(\frac{t}{\tau_g}\right)\left[\frac{c(t)}{c(0)}\right]^2 dt\right\}^2}} \tag{3.5-3}$$

and τ_g is the duration of a typical gate pulse.

The procedures described above are followed to develop the second, or remaining, part of (3.4-5) needed in (3.3-11). We shall omit the details; the final result is

$$\text{Second part of } \frac{2\pi T_R}{V_\Sigma^2\, \bar{d}_1^2} \int_{-\infty}^{\infty} \mathcal{A}\{R_V(t,t+\tau)\}\, dt$$

$$= \frac{1}{(S_1/N)} \sum_{n=0}^{\infty} \sum_{m=0}^{[n]/2} \sum_{\ell=0}^{[n]/2} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} K_{nm\ell rp}\, \bar{R}_{Bnm\ell rp} \left(\frac{S_1}{N}\right)^{n-m-\ell+r+p} \tag{3.5-4}$$

where

$$K_{nm\ell rp} \triangleq \frac{2^{n-2m-2\ell+1}\,(-1)^{m+\ell+r+p}\,(n!)^2\, B_{nm\ell rp}}{(n+1)!\; m!\; (n-2m)!\; \ell!\; (n-2\ell)!\; r!\; p!} \tag{3.5-5}$$

$$B_{nm\ell rp} \triangleq \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \cos^{n+2}(\theta_1) \int_{-\pi}^{\pi} \cos^{n-2m+2r}(\theta)\, \cos^{n-2\ell+2p}(\theta+\theta_1)\, d\theta\, d\theta_1 \tag{3.5-6}$$

$$\bar{R}_{Bnm\ell rp} \triangleq \frac{\displaystyle\int_{-\infty}^{\infty}\left[\frac{c(\tau)}{c(0)}\right]^{n+2} \int_{-T_R/2}^{T_R/2} \mathrm{rect}\left(\frac{t}{\tau_g}\right)\left[\frac{c(t)}{c(0)}\right]^{n-2m+2r} \mathrm{rect}\left(\frac{t+\tau}{\tau_g}\right)\left[\frac{c(t+\tau)}{c(0)}\right]^{n-2\ell+2p} dt\, d\tau}{\left\{\displaystyle\int_{-T_R/2}^{T_R/2} \mathrm{rect}\left(\frac{t}{\tau_g}\right)\left[\frac{c(t)}{c(0)}\right]^2 dt\right\}^2} \tag{3.5-7}$$

3.6 Accuracy

Finally, we substitute (3.5-4) and (3.5-1) into (3.3-11) to obtain the system's angle tracking error variance

$$\sigma^2(\hat{x}) = \sigma^2_{min}(\hat{x})\; R_A \left[1 + \frac{R_B}{(S_1/N)}\right] \tag{3.6-1}$$

where

$$R_A \triangleq \frac{\displaystyle\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} H_{ik}\; \bar{R}_{Aik}\left(\frac{S_1}{N}\right)^{i+k}}{\left\{\displaystyle\sum_{k=0}^{\infty} R_{ck}\left(\frac{S_1}{N}\right)^{k}\right\}^2} \tag{3.6-2}$$

$$R_B \overset{\Delta}{=} \frac{\sum_{n=0}^{\infty} \sum_{m=0}^{[n]/2} \sum_{\ell=0}^{[n]/2} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} K_{nm\ell rp} \bar{R}_{Bnm\ell rp}\left(\frac{S_1}{N}\right)^{n-m-\ell+r+p}}{\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} H_{ik} \bar{R}_{Aik} \left(\frac{S_1}{N}\right)^{i+k}} . \quad (3.6\text{-}3)$$

The main result is (3.6-1). This result has exactly the same form as the tracking error variance for other monopulse [1] and conopulse [6] radars. Therefore, the presence of the sum-channel limiter has not altered the form of the accuracy formula but the coefficients R_A and R_B have become much more complicated. In fact, they are now functions of signal-to-noise ratio in complicated series form.

3.7 Coefficient Evaluation

In many cases, as in this study, it is convenient to determine the integrals involved in the expressions (3.5-5) and (3.5-2) for coefficients $K_{nm\ell rp}$ and H_{ik}, respectively, by numerical integration. However, it is possible to develop series solutions for these coefficients, as is done in this section.

We begin by noting that both coefficients involve an integral of the form

$$Q \overset{\Delta}{=} \int_{-\pi}^{\pi} \cos^{I}(\theta) \cos^{K}(\theta + \theta_1) d\theta \quad (3.7\text{-}1)$$

where $I+K$ = even integer. We first use trigonometric identities and the binomial series as follows

$$Q = \int_{-\pi}^{\pi} \cos^{I}(\theta) \left[\cos(\theta_1) \cos(\theta) - \sin(\theta_1) \sin(\theta)\right]^{K} d\theta$$

$$= \int_{-\pi}^{\pi} \cos^{I}(\theta) \left[\sum_{r=0}^{k} (-b)^r \sin^r(\theta)\, a^{k-r} \cos^{k-r}(\theta) \binom{k}{r}\right] d\theta$$

$$= \sum_{r=0}^{k} \binom{k}{r} a^{k-r} (-b)^r \int_{-\pi}^{\pi} \cos^{I+K-r}(\theta) \sin^r(\theta)\, d\theta \quad (3.7\text{-}2)$$

where

$$a \stackrel{\Delta}{=} \cos(\theta_1) \qquad (3.7\text{-}3)$$

$$b \stackrel{\Delta}{=} \sin(\theta_1). \qquad (3.7\text{-}4)$$

Since $\cos^{I+K-r}(\theta)$ is even for all I+K-r while $\sin^r(\theta)$ is odd for r odd the integral is zero for r odd. The integral is therefore nonzero for only even values of r. Let [K] = largest even integer not exceeding K and define

$$r = 2k, \quad 0 \le k \le [K]/2. \qquad (3.7\text{-}5)$$

Then,

$$Q = \sum_{k=0}^{[K]/2} \binom{K}{2k} a^{K-2k}(-b)^{2k} \int_{-\pi}^{\pi} \cos^{I+K-2k}(\theta) \sin^{2k}(\theta) d\theta. \qquad (3.7\text{-}6)$$

Because

$$\sin^{2k}(\theta) = [1-\cos^2(\theta)]^k = \sum_{n=0}^{k} \binom{k}{n} [-\cos^2(\theta)]^{k-n} \qquad (3.7\text{-}7)$$

(3.7-6) becomes

$$Q = \sum_{k=0}^{[K]/2} \binom{K}{2k} a^{K-2k}(-b)^{2k} \sum_{n=0}^{k} \binom{k}{n} \int_{-\pi}^{\pi} (-1)^{k-n} [\cos(\theta)]^{I+K-2n} d\theta. \qquad (3.7\text{-}8)$$

The power I+K - 2n is even for all I, K, and n so a series expansion for $[\cos(\theta)]^{I+K-2n}$ can be substituted [7, p. 25] to evaluate the integral easily. We have

$$\cos^{2m}(\theta) = \frac{1}{2^{2m}} \left\{ \binom{2m}{m} + 2 \sum_{q=0}^{m-1} \binom{2m}{q} \cos[2(m-q)\theta] \right\} \qquad (3.7\text{-}9)$$

so (3.7-8) becomes

$$Q = \sum_{k=0}^{[K]/2} \binom{K}{2k} a^{K-2k} b^{2k} \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} \frac{2\pi}{2^{I+K-2n}} \binom{I+K-2n}{\frac{I+K-2n}{2}} \qquad (3.7\text{-}10)$$

or

$$Q = 2\pi \sum_{k=0}^{[K]/2} \sum_{n=0}^{k} \binom{K}{2k} \binom{k}{n} (-1)^{k-n} \cos^{K-2k}(\theta_1) \sin^{2k}(\theta_1) \binom{I+K-2n}{\frac{I+K-2n}{2}} \frac{1}{2^{I+K-2n}} \cdot \qquad (3.7\text{-}11)$$

We apply (3.7-11) to the solution of (3.5-6). After noting that in this case

$$I = n - 2m + 2r \qquad (3.7\text{-}12)$$

$$K = n - 2\ell + 2p, \qquad (3.7\text{-}13)$$

we substitute (3.7-11) into (3.5-6) which will then contain series terms involving integrals. The integrand of a typical term is $[\cos(\theta_1)]^{2(n-\ell+p+1-k)} \cdot [\sin(\theta_1)]^{2k}$. The cosine factor is even and its power is 2 or more. The sine factor is expanded according to

$$[\sin(\theta_1)]^{2k} = [1 - \cos^2(\theta_1)]^k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \cos^{2i}(\theta_1) \qquad (3.7\text{-}14)$$

and the resulting integrand is expanded using known series [7, p.25, eq. 1.320(5.)] and integrated easily. The final result is

$$B_{nm\ell rp} = \sum_{k=0}^{[n-2\ell+2p]/2} \sum_{s=0}^{k} (-1)^{k-s} \binom{n-2\ell+2p}{2k} \binom{k}{s} \binom{2(n-m-\ell+r+p-s)}{n-m-\ell+r+p-s}$$

$$\frac{1}{4^{n-m-\ell+r+p-s}} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{i}{4^J} \binom{2J}{J}, \qquad (3.7\text{-}15)$$

where

$$J \triangleq n - \ell + p + 1 - k + i. \qquad (3.7\text{-}16)$$

Coefficients $K_{nm\ell rp}$ result readily by using (3.7-15) with (3.5-5).

To evaluate coefficient H_{ik} we use the same procedure described above. We omit details and give only the result:

$$\boxed{H_{ik} = \frac{(-1)^{i+k}}{(2i+1)i!(2k+1)k!} \sum_{r=0}^{j} \sum_{s=0}^{r} (-1)^{r-s} \binom{2j+1}{2r}\binom{r}{s}\binom{2(i+j+1-s)}{i+j+1-s} \cdot \frac{1}{4^{i+j+1-s}} \sum_{q=0}^{r} (-1)^q \binom{r}{q} \frac{1}{4^{j+1-r+q}} \binom{2(j+1-r+q)}{j+1-r+q}. \qquad (3.7\text{-}17)}$$

3.8 Accuracy Coefficients for Rectangular Pulses

If transmitted pulses are rectangular with duration τ_p and amplitude $p(0)$, pulses of form $c(t)$ will be triangular of base duration $2\,\tau_p$ and peak amplitude

$$c(0) = p^2(0)\,\tau_p \qquad (3.8\text{-}1)$$

according to (3.1-3). Thus,

$$c(t) = \begin{cases} c(0)\,[1 + (t/\tau_p)], & -\tau_p \le t \le 0 \\ c(0)\,[1 - (t/\tau_p)], & 0 \le t \le \tau_p. \end{cases} \qquad (3.8\text{-}2)$$

When (3.8-2) is substituted into (3.3-5) we obtain

$$\bar{d}_{k+1} = \begin{cases} \dfrac{2\tau_p \left[1 - \left(1 - \dfrac{\tau_g}{2\,\tau_p}\right)^{2k+3}\right]}{(2k+3)}, & \tau_g \le 2\tau_p \\[2ex] \dfrac{2\tau_p}{(2k+3)}, & \tau_g > 2\,\tau_p. \end{cases} \qquad (3.8\text{-}3)$$

From (3.3-7):

$$R_{ck}=\begin{cases}\dfrac{(-1)^k(2k)!}{(k!)^2 2^{2k}(k+1)!}\cdot\dfrac{3\left[1-\left(1-\dfrac{\tau_g}{2\tau_p}\right)^{2k+3}\right]}{(2k+3)\left[1-\left(1-\dfrac{\tau_g}{2\tau_p}\right)^3\right]}, & \tau_g \leq 2\,\tau_p \\[2em] \dfrac{(-1)^k\,(2k)!}{(k!)^2\,2^{2k}(k+1)!}\cdot\dfrac{3}{(2k+3)}, & \tau_g > 2\tau_p .\end{cases} \tag{3.8-4}$$

These functions are seen to decrease in magnitude rapidly as k increases.

For k=0, $R_{ck} = 1$. When (3.8-4) is used in the denominator of the accuracy coefficient R_A given by (3.6-2), this denominator behaves as shown in Figure 3-2, as found by computer.

At this writing other computer computations of the numerator of R_A and R_B [given in (3.6-3)] were made but due to slow convergence of the series many terms were found to be necessary for accuracy. Unfortunately, higher terms generated practical machine problems (overflows) that were unsolved as the study ended. Because of these problems further evaluation of R_A and R_B remains for future work.

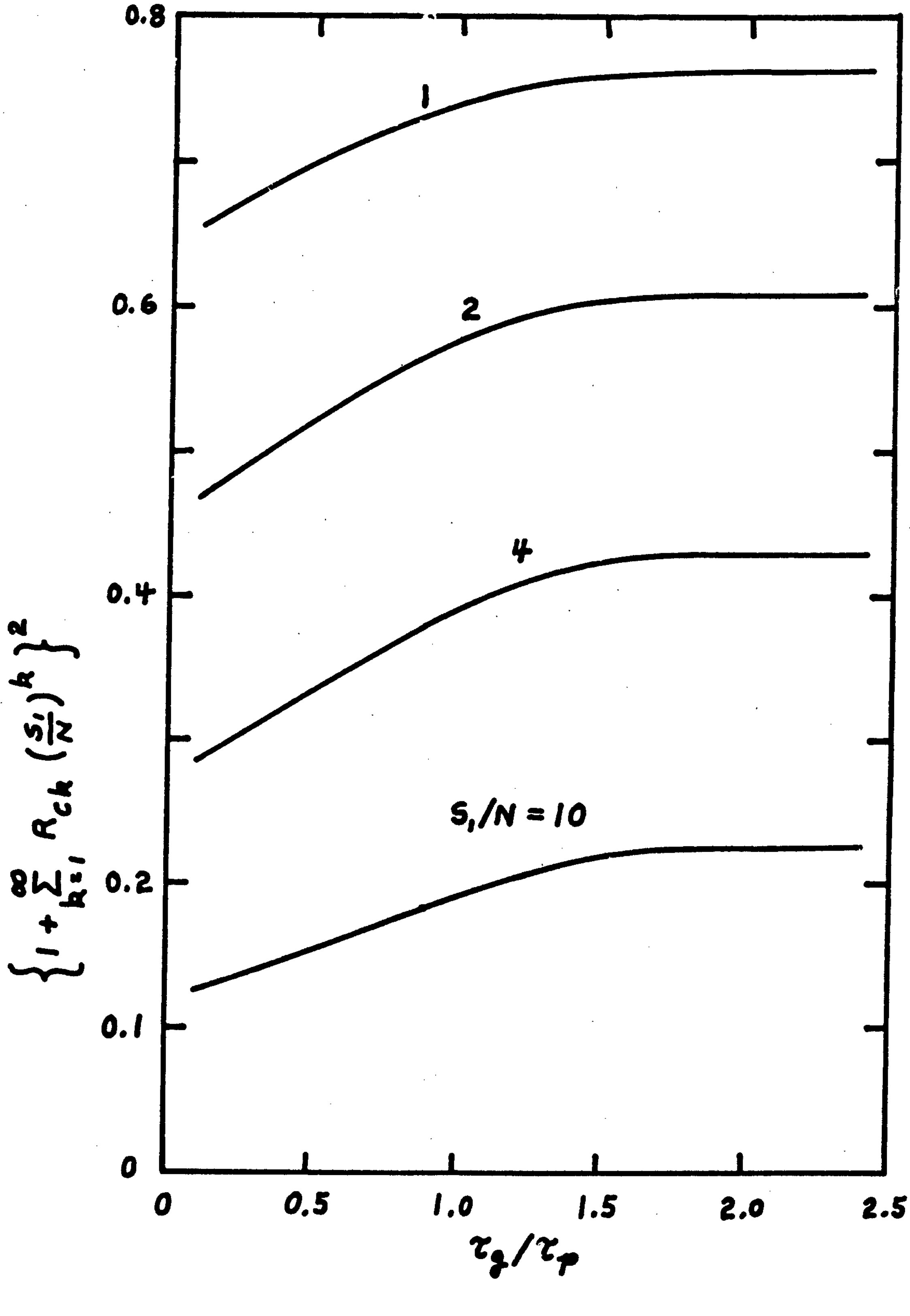

Fig. 3-2. Plots of the denominator of R_A.

4.0 ANGLE ACCURACY OF RHODES' CLASS III MONOPULSE

4.1 Introduction

In a Rhodes class III system [2] having an offset beam angle sensor (amplitude monopulse antenna), the sensor outputs are passed through a short-slot coupler to convert to a phase-sensing sensor. The coupler outputs pass to a receiver having limiters sensitive only to input phase variations. The limited outputs are then compared in an odd phase detector (one giving an output proportional to the sine of the difference of the input phases) as shown in Figure 4-1. The odd phase detector can be implemented by preceeding the usual even phase detector with a 90° phase shifter on one of its inputs.

A practical implementation of a class III system, which we shall refer to as System 6, is shown in Figure 4-2 for a single angle coordinate.

In this main section we determine an expression for the angle accuracy of the system of Figure 4-2. We begin by establishing the equivalences between outputs P_{a1} and P_{a2} for a class III system and outputs Σ and Δ of a monopulse system [1, p.3].

4.2 Amplitude Monopulse Equivalences to System 6

An amplitude monopulse antenna with its hybrid networks will produce outputs Σ, Δ_{az}, and $\Delta_{e\ell}$ from its sum and two difference channels. Within a proportionality constant that is the same for both monopulse and other systems, we may consider Σ, Δ_{az} and $\Delta_{e\ell}$ as "patterns" of the antenna.

It is convenient to think of patterns Σ, Δ_{az} and $\Delta_{e\ell}$ as generated by four space patterns A, B, C, and D. Indeed, when four feed horns

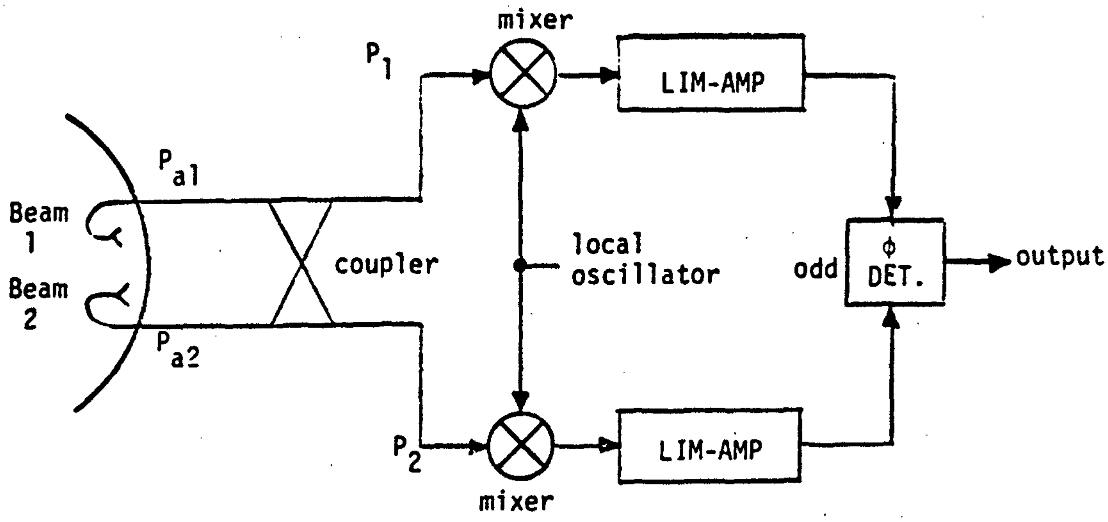

Fig. 4-1. Basic block diagram of Rhodes' Class III monopulse using an amplitude monopulse antenna.

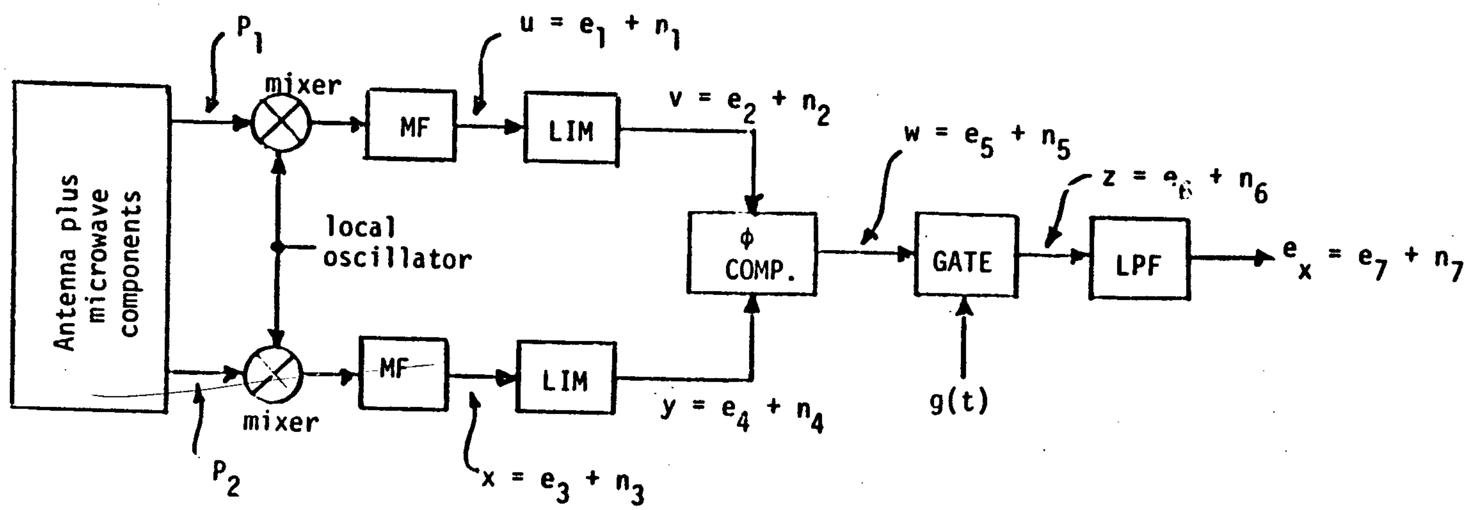

Fig. 4-2. **Practical block diagram of Rhodes' Class III monopulse receiver.**

are used with a reflector antenna, this is exactly the case as shown in Figure 4-3. Clearly,

$$\Sigma = \frac{(A+B)}{2} + \frac{(C+D)}{2} = \frac{(A+D)}{2} + \frac{(B+C)}{2} \qquad (4.2\text{-}1)$$

$$\Delta_{az} = \frac{(A+D)}{2} - \frac{(B+C)}{2} \qquad (4.2\text{-}2)$$

$$\Delta_{e\ell} = \frac{(A+B)}{2} - \frac{(C+D)}{2} \qquad (4.2\text{-}3)$$

or

$$\frac{A+B}{2} = \frac{1}{2}\left[\Sigma + \Delta_{e\ell}\right] \qquad (4.2\text{-}4)$$

$$\frac{C+D}{2} = \frac{1}{2}\left[\Sigma - \Delta_{e\ell}\right] \qquad (4.2\text{-}5)$$

$$\frac{A+D}{2} = \frac{1}{2}\left[\Sigma + \Delta_{az}\right] \qquad (4.2\text{-}6)$$

$$\frac{B+C}{2} = \frac{1}{2}\left[\Sigma - \Delta_{az}\right] \qquad (4.2\text{-}7)$$

Thus, for specified patterns Σ, Δ_{az} and $\Delta_{e\ell}$, there exist patterns in space defined by (A+B)/2, (C+D)/2, (A+D)/2 and (B+C)/2 that are combinations of space patterns A, B, C and D.

We observe that (A+B)/2 and (C+D)/2 are vertically offset patterns (in opposing directions) while (A+D)/2 and (B+C)/2 are opposingly offset in the horizontal direction. They are all exactly of the form needed to provide inputs to the class III receiver.

Next, we consider processing patterns A, B, C and D as shown in Figure 4-4. The outputs are the desired combination patterns needed in the class III system. From (4.2-4) through (4.2-7) and the definitions of P_{a1}, P_{a2}, P_{e1} and P_{e2} of Figure 4-4, we have

$$P_{a1} = \frac{1}{2}\left[\Sigma + \Delta_{az}\right] \qquad (4.2\text{-}8)$$

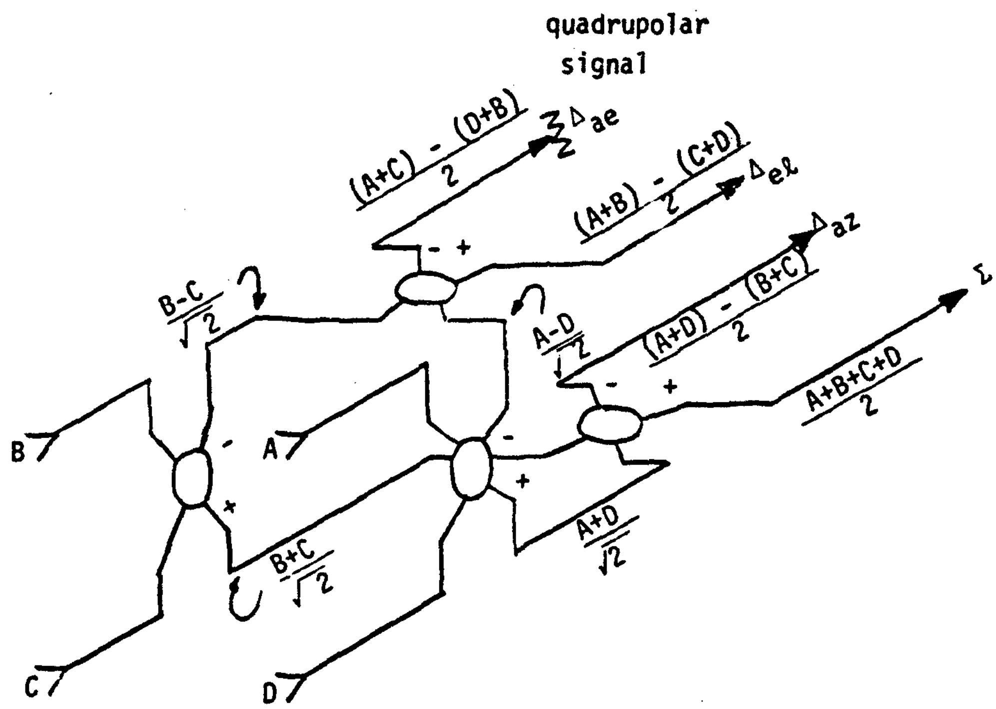

Fig. 4-3. Sum and difference hybrids for an amplitude comparison monopulse angle sensor.

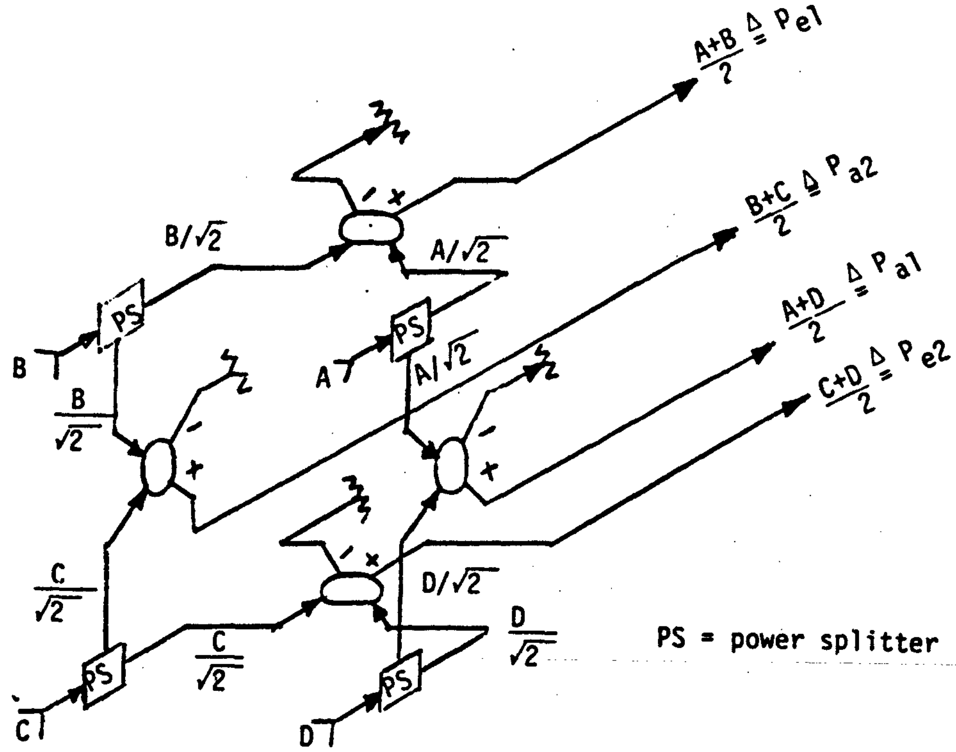

Fig. 4-4. Hybrid arrangement for use in Rhodes' Class III monopulse angle sensor.

$$P_{a2} = \frac{1}{2} [\Sigma - \Delta_{az}] \quad (4.2\text{-}9)$$

$$P_{e1} = \frac{1}{2} [\Sigma + \Delta_{e\ell}] \quad (4.2\text{-}10)$$

$$P_{e2} = \frac{1}{2} [\Sigma - \Delta_{e\ell}]. \quad (4.2\text{-}11)$$

When two of these outputs (say P_{a1} and P_{a2} corresponding to the azimuth coordinate) are processed using a short-slot coupler according to Figure 4-5, the responses are found to be

$$P_1 = \frac{1}{2} [e^{j\pi/4} \ \Sigma + e^{-j\pi/4} \ \Delta_{az}] \quad (4.2\text{-}12)$$

$$P_2 = \frac{1}{2} [e^{j\pi/4} \ \Sigma - e^{-j\pi/4} \ \Delta_{az}]. \quad (4.2\text{-}13)$$

I. summary, we have found that the responses P_1 and P_2 for a class III system using the angle sensor defined in Figures 4-4 and 4-5 are related to the responses Σ and Δ_{az} that the same antenna would produce in a sum and difference mode according to (4.2-12) and (4.2-13). A similar statement can be made for the elevation coordinate but following analysis is identical in both coordinates and only one analysis needs to be conducted.

4.3 Preliminary Analysis

In the notation of Figure 4-2 we write (2.2-14) as

$$\sigma^2(\hat{x}) = \frac{\frac{W_L}{\pi} \int_{\infty}^{\infty} \mathcal{A}\{R_Z(t, t+\tau)\} \, d\tau}{\left\langle \left. \frac{d \, \mathcal{A}\{E[Z(t)]\}}{dx} \right|_{x=0} \right\rangle^2} \quad (4.3\text{-}1)$$

where

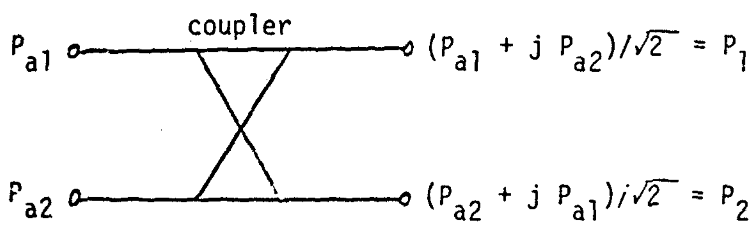

Fig. 4-5. Short slot coupler for use in Rhodes' Class III monopulse.

$$E[Z(t)] = E[g(t)W(t)] = E\,[g(t)\;V(t)\;Y(t)]$$

$$= E[g(t)\,\frac{U(t)}{|U(t)|}\;\cdot\;\frac{X(t)}{|X(t)|}\,] = g(t)E\left[\frac{U(t)}{|U(t)|}\right]E\left[\frac{X(t)}{|X(t)|}\right] \tag{4.3-2}$$

since the noises in the two channels are presumed statistically independent. From the analysis in Section 3 we know the mean values of the limiter responses [see (3.2-6)] so (4.3-2) reduces to

$$E[Z(t)] = g(t)\;\mathrm{erf}\left[\frac{e_1(t)}{\sqrt{2}\;\sigma_1}\right]\mathrm{erf}\left[\frac{e_3(t)}{\sqrt{2}\;\sigma_3}\right] \tag{4.3-3}$$

where σ_1^2 and σ_3^2 are the powers of the noises $n_1(t)$ and $n_3(t)$, respectively.

From the previous study [1] the outputs of an amplitude monopulse system are

$$\Delta_{az} = (K_o\;G_\Sigma^2\;k_m\;x/\theta_3)\sum_{n=-\infty}^{\infty} p(t-\tau_T-nT_R)\cos\left[\omega_o(t-\tau_T)+\theta_o\right] \tag{4.3-4}$$

$$\Sigma = K_o\;G_\Sigma^2\sum_{n=-\infty}^{\infty} p(t-\tau_T-nT_R)\cos\left[\omega_o(t-\tau_T)+\theta_o\right] \tag{4.3-5}$$

Where K_o, G_Σ, k_m, etc. are all defined in [1]. Next, define

$$\alpha_1 \triangleq K_o\;G_\Sigma^{\,2} \tag{4.3-6}$$

$$\alpha_2 \triangleq K_o\;G_\Sigma^{\,2}\;k_m/\theta_3 = \alpha_1\;k_m/\theta_3 \tag{4.3-7}$$

$$Per^{(1)}(t) \triangleq \sum_{n=-\infty}^{\infty} p(t-\tau_T-nT_R) \tag{4.3-8}$$

$$\theta \triangleq \omega_o(t-\tau_T)+\theta_o+(\pi/4) \tag{4.3-9}$$

so

$$\Delta_{az} = x\,\alpha_2\,Per^{(1)}(t)\cos[\theta - (\pi/4)] \quad (4.3\text{-}10)$$

$$\Sigma = \alpha_1\,Per^{(1)}(t)\cos[\theta - (\pi/4)] \quad (4.3\text{-}11)$$

and

$$P_1 = \frac{1}{2}\,Per^{(1)}(t)\,\{\alpha_1\cos(\theta) + \alpha_2\,x\sin(\theta)\} \quad (4.3\text{-}12)$$

$$P_2 = \frac{1}{2}\,Per^{(1)}(t)\,\{\alpha_1\cos(\theta) - \alpha_2\,x\sin(\theta)\} \quad (4.3\text{-}13)$$

from (4.2-12) and (4.2-13).

After the matched filters in Figure 4-2 the signals become

$$e_1(t) = \frac{1}{2}\,Per(t)\,\{\alpha_1\sin(\theta) - \alpha_2\,x\cos(\theta)\} \quad (4.3\text{-}14)$$

$$e_3(t) = \frac{1}{2}\,Per(t)\,\{\alpha_1\cos(\theta) - \alpha_2\,x\sin(\theta)\} \quad (4.3\text{-}15)$$

where

$$Per(t) \triangleq \sum_{n=-\infty}^{\infty} c(t - \tau_T - nT_R) \quad (4.3\text{-}15)$$

$$c(t) = \int_{-\infty}^{\infty} p(\xi)\,p(\xi + t)\,d\xi \quad (4.3\text{-}17)$$

$$c(t) \longleftrightarrow |P(\omega)|^2 \quad (4.3\text{-}18)$$

$$p(t) \longleftrightarrow P(\omega). \quad (4.3\text{-}19)$$

Signals $e_1(t)$ and $e_3(t)$ given by (4.3-14) and (4.3-15) will subsequently be used with (4.3-3) in (4.3-1). First, however, we compute the necessary derivative.

4.4 Derivative Calculation

Since error angle x does not involve t we can differentiate inside the time integral (time average)

$$\frac{d\,\mathcal{A}\{E[Z(t)]\}}{dx} = \mathcal{A}\left\{\frac{dE[Z(t)]}{dx}\right\} = \mathcal{A}\left\{g(t)\,\frac{d}{dx}\left\langle \operatorname{erf}\left[\frac{e_1(t)}{\sqrt{2}\,\sigma_1}\right]\operatorname{erf}\left[\frac{e_3(t)}{\sqrt{2}\,\sigma_3}\right]\right\rangle\right\}$$

$$= \mathcal{A}\left\{g(t)\left\langle \operatorname{erf}\left[\frac{e_1(t)}{\sqrt{2}\,\sigma_1}\right]\frac{d}{dx}\operatorname{erf}\left[\frac{e_3(t)}{\sqrt{2}\,\sigma_3}\right]\right.\right.$$

$$\left.\left. + \operatorname{erf}\left[\frac{e_3(t)}{\sqrt{2}\,\sigma_1}\right]\frac{d}{dx}\left[\operatorname{erf}\frac{e_1(t)}{\sqrt{2}\,\sigma_1}\right]\right\rangle\right\}. \qquad (4.4\text{-}1)$$

But

$$\frac{d\,\operatorname{erf}(u)}{du} = \frac{2}{\sqrt{\pi}}\,e^{-u^2} \qquad (4.4\text{-}2)$$

$$\frac{d\,e_1(t)}{dx} = -\,(\alpha_2/2)\cos(\theta)\,\mathrm{Per}(t) \qquad (4.4\text{-}3)$$

$$\frac{de_3(t)}{dx} = -\,(\alpha_2/2)\sin(\theta)\,\mathrm{Per}(t) \qquad (4.4\text{-}4)$$

and, if we assume

$$\sigma_1 = \sigma_3 \overset{\Delta}{=} \sigma \qquad (4.4\text{-}5)$$

then

$$\frac{d\,\mathcal{A}\{E[Z(t)]\}}{dx} = \frac{-\alpha_2}{\sqrt{2\pi\sigma^2}}\,\mathcal{A}\left\{g(t)\,\mathrm{Per}(t)\left\langle \sin(\theta)\,e^{-e_3^2/2\sigma^2}\operatorname{erf}\left[\frac{e_1}{\sqrt{2}\,\sigma}\right]\right.\right.$$

$$\left.\left. + \cos(\theta)\,e^{-e_1^2/2\sigma^2}\operatorname{erf}\left[\frac{e_3}{\sqrt{2}\,\sigma}\right]\right\rangle\right\}. \qquad (4.4\text{-}6)$$

After substituting the series expansions for the error function and the exponential (4.4-6) can be reduced to

$$\frac{d\,\mathcal{A}\{E[Z(t)]\}}{dx}\bigg|_{x=0} = \frac{-\sqrt{2}\;\alpha_2}{\pi\sigma} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \alpha_1^{2k+2\ell+1}}{\ell!(2\sqrt{2}\;\sigma)^{2k+1}(8\sigma^2)^\ell}$$

$$\cdot\, \mathcal{A}\{g(t)[Per(t)]^{2k+2\ell+2}[\sin^{2k+2}(\theta)\cos^{2\ell}(\theta)+\cos^{2k+2}(\theta)\sin^{2\ell}(\theta)]\}. \tag{4.4-7}$$

The factor involving sines and cosines varies rapidly with time compared with other time factors. The time average will then very nearly equal the time average of the other factors multiplied by the constant (in t) part of the sine/cosine factor.

Define

$$C_{ok\ell} \triangleq \text{constant part of } [\sin^{2k+2}(\theta)\cos^{2\ell}(\theta) + \cos^{2k+2}(\theta)\sin^{2\ell}(\theta)] \tag{4.4-8}$$

and note that in the equivalent amplitude monopulse system the signal-to-noise ratio at the matched filter output would be

$$\frac{S_1}{N} = \frac{\alpha_1{}^2 c^2(0)}{2\,\sigma^2}\,. \tag{4.4-9}$$

By using (4.4-8) and (4.4-9) in (4.4-7) we get

$$\frac{d\,\mathcal{A}\{E[Z(t)]\}}{dx}\bigg|_{x=0} = \frac{-k_m(S_1/N)}{\pi\theta_3} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell C_{ok\ell}}{\ell!\,2^{2k+2\ell}}$$

$$\cdot\left(\frac{S_1}{N}\right)^{k+\ell} \mathcal{A}\left\{g(t)\left[\frac{Per(t)}{c(0)}\right]^{2k+2\ell+2}\right\}. \tag{4.4-10}$$

4.5 Accuracy

If we define

$$M_{k\ell} \triangleq \frac{C_{ok\ell}\,(-1/4)^{k+\ell}}{k!\,(2k+1)\,\ell!} \tag{4.5-1}$$

$$\bar{R}_{Ck\ell} \triangleq \mathcal{A}\left\{g(t)\left[\frac{Per^2(t)}{c^2(0)}\right]^{k+\ell+1}\right\}. \tag{4.5-2}$$

and use (3.3-9) with (4.4-10) we can write (4.3-1) as

$$\sigma^2(\hat{x}) = \sigma^2_{min}(\hat{x})\,\frac{\frac{2\pi^2}{TR}\int_{-\infty}^{\infty}\mathcal{A}\,\{R_Z(t,t+\tau)\}\,d\tau}{\left(\frac{S_1}{N}\right)\left\{\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} M_{k\ell}\bar{R}_{Ck\ell}\left(\frac{S_1}{N}\right)^{k+\ell}\right\}^2}\,. \tag{4.5-3}$$

This result requires the development of $R_Z(t, t+\tau)$. It can be shown that

$$R_Z(t,t+\tau) = g(t)g(t+\tau)\;R_V(t,t+\tau)\;R_Y(t,t+\tau). \tag{4.5-4}$$

However, the study terminated before the product $R_V(t,t+\tau)\;R_Y(t,t+\tau)$ could be solved. Final solution of (4.5-3) remains for future work.

4.6 Coefficient Evaluation

Although we shall omit the proofs, it can be shown that

$$C_{ok\ell} = \frac{1}{2^{2k+1}}\sum_{n=0}^{\ell}\binom{\ell}{n}(-1/4)^n\binom{2k+2n+2}{k+n+1} \tag{4.6-1}$$

Also, another form is

$$C_{ok\ell} = \frac{1}{2^{2\ell-1}}\sum_{n=0}^{k+1}\binom{k+1}{n}(-1/4)^n\binom{2\ell+2n}{\ell+n} \tag{4.6-2}$$

Two other relationships that can be proved from these two equations, that are not previously known to the author, are

$$\sum_{n=0}^{\ell} \binom{\ell}{n}\binom{2n+2}{n+1}\left(\frac{-1}{4}\right)^{n} = \frac{1}{2^{2\ell}}\binom{2\ell}{\ell}\frac{2}{(\ell+1)} \qquad (4.6\text{-}3)$$

$$\sum_{n=0}^{\ell} \binom{\ell}{n}\binom{2n}{n}\left(\frac{-1}{4}\right)^{n} = \frac{1}{2^{2\ell}}\binom{2\ell}{\ell}\,. \qquad (4.6\text{-}4)$$

5.0 SUMMARY AND DISCUSSION

This study has directed its efforts in four principal areas. The first was to rigorously determine the angle tracking accuracy of an amplitude comparison monopulse radar having a limiter in the sum channel prior to the phase (angle) detector. This effort was successfully completed and accuracy is determined through the variance of angle error in (3.6-1). The accuracy expression (3.6-1) requires computation of accuracy coefficients R_A and R_B, which are given in general by (3.6-2) and (3.6-3). These, in turn, require series coefficients H_{ik} and $K_{mn\ell rp}$ as well as series functions $\bar{R}_{Aik}$, R_{ck} and $\bar{R}_{Bnm\ell rp}$; these are all given by (3.5-2), (3.5-5) using (3.5-6), (3.5-3), (3.3-7) using (3.3-5), and (3.5-7), respectively. All these results are valid for any shaped transmitted signal, and any duration range gate (range gate is, however, limited to a rectangular shape, where the receiver is assumed to have matched filters).

The second effort area was to derive a general expression for the angle tracking accuracy of any system, whether it be linear, nonlinear or have any form of signal processor. The main result is (2.2-12), which applies even for off-boresight tracking where $x_0 \neq 0$. For boresight tracking under the special (but usually satisfied) constraint where the time-averaged mean output of the system is an odd function of off-boresight angle x, the applicable result is (2.2-14). The result (2.2-14) was applied in the analysis described in the foregoing paragraph.

The third area of effort centered around numerical evaluation of the accuracy coefficients R_A and R_B for the amplitude monopulse system having a limiter and using rectangular transmitted pulses. This effort

was only partially completed. Practical computer problems prevented evaluation of all but the denominator of R_A, which is plotted in Figure 3-2 for several values of single pulse signal-to-noise ratio.

The fourth main area of effort during the study was to find the general accuracy of a Rhodes' Class III monopulse angle tracking system. This work was taken as far as possible by the end of the study. The interim result is (4.5-3). Future work is needed to obtain a final variance expression.

In addition to the four main work areas some effort was made to reduce the 5-dimensional sum required in (3.6-3). Some progress was made but the work was incomplete at this writing. Other work was also done to model generally the ideal limiter; this work was too incomplete to include herein.

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